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Excitation of internal waves by a turbulent boundary layer

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In a barotropic fluid, a free turbulent flow induces a fluctuating potential flow which is determined by the instantaneous flow near the edge of the turbulent flow. If the surrounding fluid is stably stratified, internal wave-motions are possible and, in general, wave-energy accumulates until it is sufficient to modify the turbulent flow. Here the growth of wave-motion from rest is examined with particular reference to the atmospheric problem of wave excitation by the surface boundary layer. Wind shear is supposed negligible outside the turbulent flow and the disturbances from the boundary layer are assumed to travel with a convection velocity **V** relative to the upper air. For times large compared with $\{-g/\rho(d\rho/dz)\}^{-\frac{1}{2}}$ (ρ is the potential density), most of the wave-energy resides in components of phase-velocity near the convection velocity. For a model atmosphere with increased stability above a tropopause, this resonance mechanism leads to the formation of wave-groups with crests at right-angles to the convection velocity and wavelengths near $2\pi V[-g/\rho(d\rho/dz)]^{-\frac{1}{2}}$. Using likely values for the surface disturbances, wave-amplitudes of order 100 m can develop within several hours of the initiation of the boundary layer but sufficient amplitude to produce overturning or breaking is unlikely within a reasonable time.

1. Introduction

Observations of clear-air turbulence and of shadow bands cast by starlight show that turbulent mixing is frequently intense near the tropopause. While some of the observed turbulence is associated with the presence of jet-streams, some is not and may arise from breaking of internal waves propagating in the atmosphere. One possible source of energy for internal waves is the boundary layer on the earth's surface and the main purpose of this paper is to find the probable magnitudes and characteristics of wave-motions excited by a turbulent boundary layer at the bottom of a stably stratified atmosphere. The results may have an application to the ocean.

The problem of the induced motion outside a free turbulent flow has been treated by Phillips (1955) for a barotropic fluid. The induced flow in a barotropic fluid is a potential flow and is determined by the instantaneous turbulent motion. Its energy is effectively part of the turbulent energy and its influence on the turbulent motion could be described as the addition of virtual mass to the eddies. If the surrounding fluid is stably stratified, internal wave-motions are possible and the turbulent flow can lose energy by radiation. A statistically steady state is possible if the turbulent flow is horizontally homogeneous and time-independent and can take two extreme forms. If energy dissipation outside the boundary layer is negligible, the wave-motion will grow in intensity until it transfers energy to the boundary layer at the same rate as the layer radiates. On the other hand, if dissipation outside the layer is sufficient to absorb all the radiated energy while the wave-motion is comparatively weak, the motion in the boundary layer will be nearly independent of the waves and the equilibrium wave-intensity depends on a balance between the rate of radiation and the loss by frictional forces. For large values of the wave Reynolds number, the wave-motions are nearly the same as in an inviscid fluid and the rate of wave-growth in the inviscid case will equal the rate of radiation in the dissipative case. The basic problem is thus the growth of wave-motion in a stably-stratified fluid, initially wave-free but subjected to specified displacements at its lower boundary.

2. Free and forced wave-motions

If the boundary layer is thin compared with the depth of the atmosphere, its effect on the atmosphere as a whole can be approximated by the effect of the vertical displacements which exist just outside the layer, and these displacements may be considered to occur at ground level. The instantaneous pattern of the surface displacement can be represented as the superposition of Fourier components, and the time-variation is nearly equivalent to having a pattern that is convected with uniform velocity but undergoes considerable change in times comparable with typical time-scales of the turbulent motion. The problem reduces to that of discovering the response of an initially wave-free atmosphere to a suddenly imposed, sinusoidal surface-displacement travelling with the convection velocity, and then of superimposing the responses for different phases and wave-numbers. For the comparatively small wave-numbers that are likely to excite internal waves, the convection velocity is nearly independent of wave-number.

Consider an atmosphere without wind-shear and with a continuous distribution of potential density ρ defined by

$$\beta(z) = -\frac{g}{\rho}\frac{d\rho}{dz} \quad \text{and} \quad \rho = 0 \quad \text{for} \quad z > h+H.$$

The top to the atmosphere at $z = h+H$ (the total depth is written in this way to simplify later equations for a two-layer model) reflects the finite mass per unit area of the atmosphere and does not impose a characteristic length-scale on the wave-motions to be considered. A travelling wave with small vertical displacement,

$$\zeta_1 = \psi(z)\exp\{i(\mathbf{k}.\mathbf{r}-kct)\},$$

satisfies the inviscid equations of motion (Lamb 1932) if $\psi(z)$ satisfies

$$\frac{d^2\psi}{dz^2} = \left(k^2 - \frac{\beta(z)}{c^2}\right)\psi. \tag{2.1}$$

Here $\mathbf{k}$ is the horizontal wave-number vector ($|\mathbf{k}| = k$), $\mathbf{r}$ is the horizontal position vector, c is the phase velocity, and the frame of reference is at rest with respect to the atmosphere. The boundary condition at the upper free surface is that $\psi(h+H) = c^2/g\,\psi'(h+H)$ (Lamb 1932), which approximates to $\psi(h+H) = 0$, the condition for a rigid boundary, if the phase velocities are small compared with $[g(h+H)]^{\frac{1}{2}}$, which is nearly the velocity of sound in an analogous real atmosphere. The lower boundary condition for free waves is that $\psi(0) = 0$, but a ground disturbance with displacement

$$\zeta_1 = a_0 \exp\{i(\mathbf{k}.\mathbf{r} - kct)\}$$

produces a forced wave with ψ satisfying equation (2.1), the upper boundary condition, and the lower boundary condition $\psi(0) = a_0$.

The general motion for a surface disturbance of a single wave-number $\mathbf{k}$ moving with the convection velocity $\mathbf{V}$ relative to the atmosphere is a linear combination of all the free waves of wave-number $\mathbf{k}$ and the forced wave with the appropriate surface displacement and phase-velocity. For a motion started from rest at zero time by the sudden appearance of the surface displacement, the particle displacements are

$$\zeta = a_0 \exp\{i\mathbf{k}.\mathbf{r}\} \sinh k(h+H-z)/\sinh k(h+H),$$

the same as for irrotational flow. For large values of $k(h+H)$, the displacements are very nearly

$$\zeta = a_0 \exp\{-kz\} \exp\{i\mathbf{k}.\mathbf{r}\}.$$

At finite times, the phases of the free waves have changed with respect to the forced wave and the displacement pattern is no longer irrotational. The displacements at heights for which $\exp\{-kz\}$ is very small arise from these phase changes.

Writing the forced wave as

$$\psi_1(z) \exp\{i(\mathbf{k}.\mathbf{r} - \mathbf{k}.\mathbf{V}t)\},$$

and the free waves as

$$\sum_i a_i \psi_i(z) \exp\{i(\mathbf{k}.\mathbf{r} - kc_i t)\},$$

the initial condition is satisfied if

$$a_0 \exp\{-kz\} = \psi_1(z) + \sum_i a_i \psi_i(z). \tag{2.2}$$

The subscript i refers to the various wave-modes with horizontal wave-number k. From the differential equation for $\psi(z)$ and the boundary conditions, it is easily shown that

$$\int_0^{h+H} \beta(z)\, \psi_i(z)\, \psi_j(z)\, dz = 0 \quad \text{unless} \quad i = j$$

and that

$$\int_0^{h+H} \beta(z)\, \psi_1(z)\, \psi_i(z)\, dz = a_0 \left(\frac{d\psi_i}{dz}\right)_{z=0} \frac{c_i^2\, V^2 \cos^2\theta}{V^2 \cos^2\theta - c_i^2}.$$

It follows that

$$\frac{a_i}{a_0} = \frac{\displaystyle\int_0^{h+H} \beta(z)\, e^{-kz}\, \psi_i(z)\, dz + \frac{c_i^2\, V^2 \cos^2\theta}{c_i^2 - V^2 \cos^2\theta} \left(\frac{d\psi_i}{dz}\right)_{z=0}}{\displaystyle\int_0^{H+h} \beta(z)\, \psi_i^2(z)\, dz}, \tag{2.3}$$

giving the amplitudes of the free waves in terms of the surface displacement. At time t, the spatial amplitude of the disturbance at height z is

$$\begin{aligned}\zeta(z,t) &= \psi_1(z)\exp\{-i\mathbf{k}.\mathbf{V}t\}+\sum_i a_i\psi_i(z)\exp\{-ikc_it\}\\ &= -a_0\exp\{-kz\}\exp\{-i\mathbf{k}.\mathbf{V}t\}\\ &\quad +2i\sum_i a_i\psi_i(z)\sin[\tfrac{1}{2}kt(V\cos\theta-c_i)]\exp\{-\tfrac{1}{2}ikt(c_i+V\cos\theta)\},\end{aligned} \tag{2.4}$$

where θ is the angle between the wave-number $\mathbf{k}$ and the convection velocity $\mathbf{V}$. Except very near the ground, the motion is nearly the sum of 'free' waves travelling with the modified phase velocities, $\frac{1}{2}(c_i+V\cos\theta)$, and beating with frequencies $k(c_i-V\cos\theta)$. It is clear from equation (2.3) that the most strongly excited modes are those for which $k(c_i-V\cos\theta)$ is small; these beat slowly and reach their maximum amplitude only after a considerable time. The situation is similar to that discussed by Phillips (1957) in his theory of surface waves generated by turbulent pressure fluctuations.

If the variation of the stability parameter β is small within a wavelength of the surface, $\beta(z)\,\psi_i(z)\exp\{-kz\}$ may be approximated by

$$\beta_1 e^{-kz}\left(\frac{d\psi_i}{dz}\right)_{z=0}\left(\frac{\beta_1}{c_i^2}-k^2\right)^{-\frac{1}{2}}\sin\left[\left(\frac{\beta_1}{c_i^2}-k^2\right)^{\frac{1}{2}}z\right],$$

where β_1 is the value of β near $z=0$, and equation (2.3) for the wave-amplitudes becomes

$$\frac{a_i}{a_0}=\frac{c_i^4}{c_i^2-V^2\cos^2\theta}\left(\frac{d\psi_i}{dz}\right)_{z=0}\Big/\int_0^{h+H}\beta(z)\,\psi_i^2(z)\,dz. \tag{2.5}$$

So far it has been assumed that the ground disturbance preserves its complex amplitude, which is likely to be true only for short intervals of time. If the amplitude varies with time, the displacement pattern can be expressed as the sum of the patterns caused by successive short intervals of ground disturbance, each with the current amplitude. The solution above refers to a disturbance zero for $t<0$ and equal to $a_0\exp\{i(\mathbf{k}.\mathbf{r}-kVt\cos\theta)\}$ for $t>0$. By superimposing a second disturbance which is zero for $t<\delta t$ and $-a_0\exp\{i(\mathbf{k}.\mathbf{r}-kVt\cos\theta)\}$ for $t>\delta t$, we obtain the displacement pattern caused by a short interval of disturbance. For $t>\delta t$, the two forced waves have opposite signs and the pattern consists of the two sets of free waves. For each mode, the amplitudes have the same magnitude and the relative phase differs from π by the phase advance relative to the disturbance undergone by the first wave in the interval δt. That is, the resultant amplitude of one mode for $t>\delta t$ is

$$a_i[1-\exp\{ik\delta t(c_i-V\cos\theta)\}]\approx -ika_i(c_i-V\cos\theta)\,\delta t$$

for small δt, with a_i given by equation (2.5). In any small interval of time, internal waves with mode amplitudes determined by the current value of a_0 are added to the wave-system and travel relative to the convection velocity at a speed $\Delta c_i=c_i-V\cos\theta$. It follows that the total amplitude after a finite time t is

$$\begin{aligned}\zeta = -ik\sum_i\psi_i(z)\,\Delta c_i\left(\frac{a_i}{a_0}\right)\int_0^t a_0(t')\exp\{ik\Delta c_i(t-t')\}\,dt'&\\ +a_0(t)\,e^{-kz}\exp(-ikVt\cos\theta),&\end{aligned} \tag{2.6}$$

the last term representing the effect of the instantaneous surface displacement. Introducing ensemble averages to find the expected motion after a finite time,

$$\langle b_i b_i^* \rangle = 2k^2 \Delta c_i^2 \left(\frac{a_i}{a_0}\right)^2 \langle a_0 a_0^* \rangle \int_0^t (t-\tau)\, R(\tau) \cos(k\Delta c_i \tau)\, d\tau, \tag{2.7}$$

where b_i is the amplitude of the ith mode, and

$$R(\tau) = \frac{\langle a_0(t)\, a_0^*(t+\tau) + a_0^*(t)\, a_0(t+\tau) \rangle}{2\langle a_0(t)\, a_0^*(t) \rangle}$$

is the auto-correlation function for disturbance components of wave-number k in a co-ordinate system moving with the convection velocity. For elapsed times long compared with the time scale,

$$\tau_0 = \int_0^\infty R(\tau)\, d\tau,$$

$$\langle b_i b_i^* \rangle = 2k^2 \Delta c_i^2 \left(\frac{a_i}{a_0}\right)^2 \langle a_0 a_0^* \rangle\, t \int_0^\infty R(\tau) \cos(k\Delta c_i \tau)\, d\tau. \tag{2.8}$$

The integral approaches τ_0 for small values of $k\Delta c_i \tau_0$ and decreases to zero as $k\Delta c_i \tau_0$ becomes large. Since

$$\Delta c_i^2 \left(\frac{a_i}{a_0}\right)^2 = \frac{c_i^8}{(c_i + V\cos\theta)^2} \left(\frac{d\psi_i}{dz}\right)_0^2 \Big/ \left[\int_0^{h+H} \beta\, \psi_i^2\, dz\right]^2$$

does not vary rapidly with c_i, free waves are excited with appreciable amplitudes only if $k\Delta c_i \tau_0$ is not large and with mode amplitudes about

$$2k^2 \left(\Delta c_i \frac{a_i}{a_0}\right)^2 \langle a_0 a_0^* \rangle\, t\tau_0.$$

The behaviour of the auto-correlation function may be represented roughly by supposing that $R(\tau) = 1$ for $|\tau| < \tau_0$ and is zero for larger values of $|\tau|$. Then the expected wave-motion for large values of t/τ_0 is statistically similar to the motion at $t = \tau_0$ calculated on the basis of no change in the amplitude or phase of the ground displacement but with energy increased by a factor of t/τ_0.

For ground displacements, described by a power spectrum $F(\mathbf{k})$, the power spectrum of vertical displacements at height z and time t after the start is

$$G(\mathbf{k}) = F(\mathbf{k}) \left| \sum_i A_i \psi_i(z) \frac{c_i^4}{c_i^2 - V^2\cos^2\theta} (1 - \exp\{ik\tau_0 \Delta c_i\}) \right|^2 \frac{t}{\tau_0}, \tag{2.9}$$

where
$$A_i = \left(\frac{d\psi_i}{dz}\right)_0 \Big/ \int_0^{h+H} \beta\, \psi_i^2\, dz$$

and it has been assumed that the time-scale τ_0 is the same for all wave-numbers.

3. Waves in a model atmosphere

We now consider waves excited in the density distribution defined by

$$\beta(z) = \beta_1 \quad \text{for} \quad 0 < z < h$$
$$\beta(z) = \beta_2 \quad \text{for} \quad h < z < h+H,$$

with β_1 less than β_2. Approximations will be made that refer to an ideal atmosphere with a lapse rate of 7 deg km^{-1} up to a tropopause, represented by parameters with the orders of magnitude, $\beta_1 = 2\times 10^{-4}\,\text{sec}^{-2}$, $\beta_2 = 5\times 10^{-4}\,\text{sec}^{-2}$, $h = H = 8\,\text{km}$. Examination of equation (2.1) and the boundary conditions for $\psi(z)$ shows that possible free waves are of two kinds:

(i) 'Exponential' waves with phase velocities between $\beta_1^{\frac{1}{2}}k^{-1}$ and $\beta_2^{\frac{1}{2}}k^{-1}$.

(ii) 'Sinusoidal' waves with phase velocities less than $\beta_1^{\frac{1}{2}}k^{-1}$.

The exponential waves have amplitude distributions

$$\left.\begin{aligned} \psi(z) &= \sin[k_2(z-h-H)] \quad \text{for} \quad h<z<h+H,\\ &= -\frac{\sin k_2 H}{\sinh k_1' h}\sin k_1' z \quad \text{for} \quad 0<z<h, \end{aligned}\right\} \tag{3.1}$$

where

$$\left.\begin{aligned} k_1'^2 = k^2-\beta_1 c^{-2}, \quad k_2^2 = \beta_2 c^{-2}-k^2\\ \text{and} \quad \frac{\tanh k_1' h}{k_1'}+\frac{\tan k_2 H}{k_2} = 0. \end{aligned}\right\} \tag{3.2}$$

If kH is large, at $z = h$

$$A_i\psi_i = -\frac{2k_1'}{\beta_2 H}\sin^2 k_2 H \operatorname{cosech} k_1' h \tag{3.3}$$

very nearly. A boundary layer of thickness small compared with h is likely to excite only the modes with large values of kh and, in general, $k_1' h$ is also large and $A_i\psi_i$ is extremely small. It follows that exponential modes are not excited with appreciable amplitude by the kind of disturbances expected in a boundary layer.

Sinusoidal modes have amplitude distributions

$$\left.\begin{aligned} \psi(z) &= \sin k_2(z-h-H) \quad \text{for} \quad h<z<h+H),\\ &= -\frac{\sin k_2 H}{\sin k_1 h}\sin k_1 z \quad \text{for} \quad 0<z<h, \end{aligned}\right\} \tag{3.4}$$

which satisfies equation (2.1) and the various boundary conditions provided

$$\left.\begin{aligned} k_1^2 = \beta_1 c^{-2}-k^2, \quad k_2^2 = \beta_2 c^{-2}-k^2\\ \text{and} \quad \frac{1}{k_1}\tan k_1 h+\frac{1}{k_2}\tan k_2 H = 0. \end{aligned}\right\} \tag{3.5}$$

For $z < h$,

$$A_i\psi_i = -\frac{k_1^2+k_2^2}{k_1(\beta_1 h+\beta_2 H)}\left[1+\frac{k_2^2-k_1^2}{k_1^2+k_2^2}\cos 2k_2 H\right]\sin k_1 z. \tag{3.6}$$

The maximum possible value of k_1^2/k_2^2 is, from (3.5), β_1/β_2, which is not large, and we may omit the second term in the bracket and use without serious error the 'average' value,

$$A_i\psi_i = -\frac{k_1^2+k_2^2}{k_1(\beta_1 h+\beta_2 H)}\sin k_1 z. \tag{3.7}$$

Substituting in equation (2.9), the spatial spectrum of the displacements at time t is (omitting exponential modes)

$$G(\mathbf{k}) = \left|\sum_s \frac{k_1^2+k_2^2}{\beta_1 h+\beta_2 H}\frac{kc_s^4}{c_s+V\cos\theta}\frac{\sin k_1 z}{k_1}\frac{\sin(\frac{1}{2}k\tau_0\Delta c_s)}{\frac{1}{2}k\Delta c_s}\right|^2\frac{t}{\tau_0}F(\mathbf{k}), \tag{3.8}$$

where c_s, k_1, k_2 refer to the sth mode of sinusoidal form with wave-number $\mathbf{k}$. The characteristic values for the different modes can be located roughly by noticing that particular solutions of the third of the conditions (3.5) are

$$k_1 h = p\pi, \quad k_2 H = q\pi,$$

and

$$k_1 h = (p+\tfrac{1}{2})\pi, \quad k_2 H = (q+\tfrac{1}{2})\pi,$$

where p, q are integers. Ignoring the fine detail, the third condition may be replaced by

$$k_1 h + k_2 H = r\pi, \tag{3.9}$$

where r is a positive integer. The first two conditions lead to

$$\frac{k_1^2}{\beta_1} - \frac{k_2^2}{\beta_2} = \left(\frac{1}{\beta_2} - \frac{1}{\beta_1}\right) k^2,$$

and, if k_1^2/k_2^2 is small, it is nearly true that

$$k_1 = (s+\alpha_s)\frac{\pi}{h} \quad (0 < \alpha_s < 1). \tag{3.10}$$

It turns out that, at heights comparable with h, the low-order modes contribute most to the wave-motion and (3.10) provides an adequate description of the distribution of values of k_1. Since

$$k_1^2 = \frac{(s+\alpha)^2\pi^2}{h^2} = \frac{\beta_1}{c_s^2} - k^2,$$

possible values of c_s are distributed densely just below $\beta_1^{\frac{1}{2}}/k$, corresponding with small values of k_1 and s.

Returning to the spectrum equation (3.8), it will be seen that the variation of the 'resonance factor', $\sin(\frac{1}{2}k\tau_0\Delta c_s)/\frac{1}{2}k\Delta c_s$, with mode number s is much less than the variation of the factor $\sin(k_1 z)/k_1$ if $\beta_1^{\frac{1}{2}}\tau_0 \ll k^2 zh$. For values of k likely to occur in the boundary layer, i.e. over $10^{-5}\,\text{cm}^{-1}$, the condition is satisfied near $z = h$ for time-scales less than 5000 sec. Substantial changes in the displacement pattern at the surface would be expected in times of order 500 sec. Then the sum over all modes can be approximated by neglecting the variation with mode order of terms other than $\sin k_1 z/k_1$ and replacing them by their values at $k_1 = 0$. For this to be possible, modes of fairly high order must exist with phase velocities near $\beta_1^{\frac{1}{2}}/k$. The necessary condition is that $\beta_1^{\frac{1}{2}}h/\pi c_s$ should be large, true if kh is large. Substituting values for $k_1 = 0$,

$$k_2 = \left(\frac{\beta_2-\beta_1}{\beta_1}\right)^{\frac{1}{2}} k, \quad c = \beta_1^{\frac{1}{2}}/k,$$

we find

$$G(\mathbf{k}) = \frac{\beta_1(\beta_2-\beta_1)^2}{(\beta_1 h+\beta_2 H)^2}\frac{k^2V^2\cos^2\theta}{(\beta_1^{\frac{1}{2}}+kV\cos\theta)^2}\left[\frac{\sin\{\frac{1}{2}\tau_0(\beta_1^{\frac{1}{2}}-kV\cos\theta)\}}{\frac{1}{2}\tau_0(\beta_1^{\frac{1}{2}}-kV\cos\theta)}\right]^2$$
$$\times\left[\sum_{s=0}^{\infty}\frac{\sin(s+\alpha)\pi z/h}{(s+\alpha)\pi z/h}\right]^2 z^2 t\tau_0 F(\mathbf{k}). \tag{3.11}$$

The series cannot be summed without details of modes but the major contribution comes from terms with s less than $\frac{1}{2}h/z$. For small values of z/h, there are many such terms, each nearly one, and the series is not significantly different from

$$\sum_1^\infty \frac{\sin(s\pi z/h)}{s\pi z/h} = \frac{h}{2z}\left(1-\frac{z}{h}\right) \quad \text{for} \quad 0 < \frac{z}{h} < 1.$$

For values of z/h near one, the major contribution is likely to come from the first term, $\sin(\alpha_0\pi z/h)/(\alpha_0\pi z/h)$, and to depend on the unspecified fraction α_0. That the remaining terms add little to the total for $z/h = 1$ can be seen from the following argument. If the various α's were distributed at random in the interval $(0, 1)$, the expected value for the square of the sum would be

$$\left\langle\left[\sum_0^\infty \frac{\sin[(s+\alpha)\pi]}{(s+\alpha)\pi}\right]^2\right\rangle = \left\langle\left(\frac{\sin\alpha_0\pi}{\alpha_0\pi}\right)^2\right\rangle + \left\langle\sum_{s=1}^\infty\left(\frac{\sin[(s+\alpha)\pi]}{(s+\alpha)\pi}\right)^2\right\rangle$$

and

$$\left\langle\left(\frac{\sin\alpha_0\pi}{\alpha_0\pi}\right)^2\right\rangle = \frac{1}{\pi}\int_0^\pi\left(\frac{\sin x}{x}\right)^2 dx = 0\cdot475.$$

Now

$$\sum_{s=1}^\infty\left(\frac{\sin[(s+\alpha)\pi]}{(s+\alpha)\pi}\right)^2$$

is zero if all the α's are zero and equal to $0\cdot057$ in the most favourable arrangement of them all equal to $\frac{1}{2}$. In general, it seems likely that the square of the sum for $z/h = 1$ exceeds $(\sin\alpha_0\pi/\alpha_0\pi)^2$ by about $0\cdot03$, and may vary between $0\cdot03$ and $1\cdot03$ depending on the phase fraction α_0. As an average over various conditions, we may take $0\cdot5$, and assume for intermediate values of z/h

$$\left(\sum_{s=0}^\infty \frac{\sin[(s+\alpha)\pi z/h]}{(s+\alpha)\pi z/h}\right)^2 = \frac{1}{4}\frac{h^2}{z^2}\left(1+\frac{z}{h}\right).$$

Inserting this value in (3.11)

$$G(\mathbf{k}) = \frac{1}{4}\frac{\beta_1(\beta_2-\beta_1)^2}{(\beta_1 h+\beta_2 H)^2}\frac{k^2V^2\cos^2\theta}{(\beta_1^{\frac{1}{2}}+kV\cos\theta)^2}\left[\frac{\sin\{\frac{1}{2}\tau_0(\beta_1^{\frac{1}{2}}-kV\cos\theta)\}}{\frac{1}{2}\tau_0(\beta_1^{\frac{1}{2}}-kV\cos\theta)}\right]^2 h(h+z)\,\tau_0 tF(\mathbf{k}). \tag{3.12}$$

The basic features of the wave-spectrum are now clear. Below the 'tropopause' at $z = h$, the spectral intensity increases somewhat with height and is proportional to the time elapsed since the initiation of the surface disturbance. Wave-energy is concentrated in wave-numbers for which, nearly, $k\cos\theta = \beta_1^{\frac{1}{2}}/V$, i.e. the component of the wave-number in the direction of the convection velocity is nearly $\beta_1^{\frac{1}{2}}/V$. The degree of concentration is measured by the ratio $\frac{1}{2}\beta_1^{\frac{1}{2}}\tau_0$, which is approximately the number of crests in a typical wave-group. The lateral extent and pattern of the crests depend on the spectrum function of the ground displacement.

4. Calculation of wave intensities

The ground displacements whose power spectrum is $F(\mathbf{k})$ are the consequences of vertical velocities near the edge of the boundary layer with power spectrum $f(\mathbf{k})$. Since all the Fourier components of interest have the same convection velocity V,

$$f(\mathbf{k}) = \left[k^2V^2\cos^2\theta - \left(\frac{d^2R(\tau)}{d\tau^2}\right)_{\tau=0}\right]F(\mathbf{k}), \tag{4.1}$$

where $R(\tau)$ is the auto-correlation function defined in §2. For an isotropic distribution of vertical velocities satisfying the continuity equation (see Batchelor 1953)

$$f(\mathbf{k}) = k^2\phi(k), \tag{4.2}$$

and now the expression for the wave spectrum (3.12) can be partly integrated in polar co-ordinates to give the mean square displacement at time t_z and height z (less than h).

$$\langle\zeta^2\rangle = \tfrac{1}{8}\pi h(h+z)\,t\,\frac{\beta_1^{\frac{3}{2}}(\beta_2-\beta_1)^2}{(\beta_1 h+\beta_2 H)^2\,V^4}\,\frac{\beta_1}{\beta_1+\tau_1^{-2}}\int_0^{2\pi}\phi\left[\frac{\beta_1^{\frac{1}{2}}}{V\cos\theta}\right]\cos^{-4}\theta\,d\theta, \tag{4.3}$$

where $\tau_1^{-2} = -[d^2R(\tau)/d\tau^2]_{\tau=0}$. It has been asumed that $\beta_1\tau_0^{\frac{1}{2}}$ is so large that only the variation of the resonance factor

$$\left[\frac{\sin\{\frac{1}{2}\tau_0(\beta_1^{\frac{1}{2}}-k\,V\cos\theta)\}}{\frac{1}{2}\tau_0(\beta_1^{\frac{1}{2}}-kV\cos\theta)}\right]^2$$

need be considered in performing the integration with respect to k. A plausible form for the spectrum function is

$$\phi(k) = \frac{16}{\pi^3}\,w_0^2\,L_0^4\exp\left(-\frac{4}{\pi}\,k^2L_0^2\right), \tag{4.4}$$

where w_0^2 is the root-mean-square velocity fluctuation and L_0 is the integral scale of the fluctuations, both just outside the turbulent layer. The rapid cut-off of the spectrum at large wave-numbers is reasonable in view of the absence of true turbulence. For the spectrum (4.4),

$$\langle\zeta^2\rangle = \frac{(2\pi)^{\frac{1}{2}}}{16}\frac{w_0^2t}{\beta_1^{\frac{1}{2}}}\left(1+\frac{z}{h}\right)\left(\frac{(\beta_2-\beta_1)\,h}{\beta_1 h+\beta_2 H}\right)^2\frac{\beta_1\tau_1^2}{1+\beta_1\tau_1^2}(X^3+X)\,e^{-\frac{1}{2}X^2}, \tag{4.5}$$

where $X^2 = 8\beta_1 L_0^2/(\pi V^2)$. For fixed w_0, the maximum possible value of $\langle\zeta^2\rangle$ is

$$\zeta_m^2 = \frac{(2\pi)^{\frac{1}{2}}}{16}(1+\sqrt{2})^{\frac{1}{2}}(2+\sqrt{2})\frac{w_0^2t}{\beta_1^{\frac{1}{2}}}\left(1+\frac{z}{h}\right)\left[\frac{(\beta_2-\beta_1)\,h}{\beta_1 h+\beta_2 H}\right]^2\frac{\beta_1\tau_1^2}{1+\beta_1\tau_1^2}\exp-\left\{\frac{1+\sqrt{2}}{2}\right\}, \tag{4.6}$$

attained for $$\beta_1 L_0^2/V^2 = \tfrac{1}{8}\pi(1+\sqrt{2}).$$

The efficiency of wave-generation falls off rapidly with increasing $\beta_1 L_0^2/V^2$, i.e. with smaller convection velocities. For small values of $\beta_1 L_0^2/V^2$ (large convection velocities),

$$\langle\zeta^2\rangle = \frac{1}{4}\left(1+\frac{z}{h}\right)\frac{w_0^2L_0t}{V}\left(\frac{(\beta_2-\beta_1)\,h}{\beta_1 h+\beta_2 H}\right)^2. \tag{4.7}$$

The waves selected by the resonance mechanism have frequencies near $\beta_1^{\frac{1}{2}}$ and the mean square of the vertical velocity is simply $\beta_1\langle\zeta^2\rangle$.

If the wave amplitude is sufficiently large, non-linear effects may cause overturning and production of regions of unstable stratification. The condition for overturning is that $\partial\zeta/\partial z < -1$ and the quantity $\langle(\partial\zeta/\partial z)^2\rangle$ calculated from the linear theory must be large if overturning is likely. Making the necessary modifications to the previous argument,

$$\left\langle\left(\frac{\partial\zeta}{\partial z}\right)^2\right\rangle = \int\left|\sum_s\frac{k_1^2+k_2^2}{\beta_1 h+\beta_2 H}\frac{kc_s^3\,V\cos\theta}{c_s+V\cos\theta}\sin k_1 z\,\frac{\sin[\frac{1}{2}k\tau_0(c_s-V\cos\theta)]}{\frac{1}{2}k(c_s-V\cos\theta)}\right|^2 \times\frac{t}{\tau_0}F(\mathbf{k})\,d\mathbf{k}\quad\text{for}\quad z<h. \tag{4.8}$$

Near $z = h$, the contributions from the various modes are not closely correlated and the square of the sum of the mode amplitudes can be replaced by the sum of their squares. For a dense distribution of modes, summation can be replaced by integration over all values of s and, for sharp resonance excitation with rapid cut-off outside the pass band, the integration need consider only the variation of the resonance factor. Then

$$\left\langle\left(\frac{\partial\zeta}{\partial z}\right)^2\right\rangle = \int \frac{1}{2}\frac{t}{\tau_0}F(\mathbf{k})\int_0^\infty \left(\frac{k_1^2+k_2^2}{\beta_1 h+\beta_2 H}\right)^2\left(\frac{kc_s^3\, V\cos\theta}{c_s+V\cos\theta}\right)^2 \times\left(\frac{\sin\left[\frac{1}{2}k\tau_0(c_s-V\cos\theta)\right]}{\frac{1}{2}k(c_s-V\cos\theta)}\right)^2 ds\,d\mathbf{k}. \quad (4.9)$$

Since $k_1 = (s+\alpha)\pi/h$ and $k^2+k_1^2 = \beta_1 c_s^{-2}$, approximately

$$k(c_s - V\cos\theta) = \beta_1^{\frac{1}{2}} - k\,V\cos\theta - \frac{\pi^2\beta_1^{\frac{1}{2}}}{k^2h^2}s^2,$$

and
$$\int_0^\infty \left(\frac{\sin\left[\frac{1}{2}k\tau_0(c_s-V\cos\theta)\right]}{\frac{1}{2}k\tau_0(c_s-V\cos\theta)}\right)^2 ds = \frac{kh}{\beta_1^{\frac{1}{4}}\tau_0(\beta_1^{\frac{1}{2}}-k\,V\cos\theta)^{\frac{1}{2}}} \quad (4.10)$$

if $\beta_1^{\frac{1}{2}} > k\,V\cos\theta$ and is zero if $\beta_1^{\frac{1}{2}} < k\,V\cos\theta$. Most of the contributions to $\langle(\partial\zeta/\partial z)^2\rangle$ comes from wave-modes with phase velocities near $V\cos\theta$ so that $k_1^2+k_2^2 = (\beta_2-\beta_1)/(V\cos\theta)^2$. Using these approximations, integration with respect to k leads to

$$\left\langle\left(\frac{\partial\zeta}{\partial z}\right)^2\right\rangle = \frac{0{\cdot}406}{8}\frac{\beta_1^2 t}{V^5 h}\left(\frac{(\beta_2-\beta_1)\,h}{\beta_1 h+\beta_2 H}\right)^2 \times\int_0^{2\pi}\frac{\phi(\beta_1^{\frac{1}{2}}/V\cos\theta)}{\cos^5\theta}d\theta \quad \left(\int_0^1 x^4(1-x)^{-\frac{1}{2}}\,dx = 0{\cdot}406\right). \quad (4.11)$$

With the spectrum function (4.4), the integration can be completed with sufficient accuracy to give

$$\left\langle\left(\frac{\partial\zeta}{\partial z}\right)^2\right\rangle = \frac{0{\cdot}406}{(2\pi)^{\frac{1}{2}}}\frac{w_0^2 t}{\beta_1^{\frac{1}{2}}L_0 h}\left(\frac{(\beta_2-\beta_1)\,h}{\beta_1 h+\beta_2 H}\right)^2[(8\pi)^{\frac{1}{2}}X^4+8X]\exp\{-\tfrac{1}{2}X^2\}. \quad (4.12)$$

For given w_0 and L_0, the maximum value is

$$1{\cdot}87\frac{w_0^2 t}{\beta_1^{\frac{1}{2}}L_0 h}\left[\frac{(\beta_2-\beta_1)\,h}{\beta_1 h+\beta_2 H}\right]^2, \quad (4.13)$$

attained for $\beta_1 L_0^2/V^2 = \frac{1}{4}\pi$ very nearly, and, for large convection velocities (small $\beta_1 L_0^2/V^2$),

$$\left\langle\left(\frac{\partial\zeta}{\partial z}\right)^2\right\rangle = \frac{0{\cdot}812}{\pi}\frac{w_0^2 t}{Vh}\left(\frac{(\beta_2-\beta_1)\,h}{\beta_1 h+\beta_2 H}\right)^2. \quad (4.14)$$

At this stage, it may be useful to recall the assumptions made in obtaining the expressions for wave amplitudes. First, it is assumed that many wave-modes exist with phase-velocities near $\beta_1^{\frac{1}{2}}/k$, that is to say, the mode-order, nearly equal to $k_1 h/\pi$ by equation (3.10), can be large for values of k_1^2 which are small compared with k^2 (see the relations of (3.5)). The condition may be written

$$\pi^2 \ll k_1^2h^2 \ll k^2h^2.$$

Secondly, the variation of k_2 with mode-order is assumed to be small. From the relations (3.5),

$$k_2^2 = \frac{\beta_2-\beta_1}{\beta_1}k^2 + \frac{\beta_2}{\beta_1}k_1^2,$$

and so k_2 is nearly constant if

$$k_1^2 h^2 \ll \frac{\beta_2-\beta_1}{\beta_2}k^2h^2.$$

Thirdly, it is assumed that resonance excitation is the dominant mechanism so that $c_s \approx V\cos\theta$ and $kV\cos\theta \approx \beta_1^{\frac{1}{2}}$. All these assumptions are justified if

(i) $\dfrac{\beta_2-\beta_1}{\beta_2}\dfrac{\beta_1 h^2}{\pi^2 V^2}$ is two orders of magnitude greater than one, and if

(ii) $\beta_1^{\frac{1}{2}}\tau_0$ is fairly large. For the atmosphere, with $\beta_1 \approx 2\times 10^{-4}\,\sec^{-2}$, $\beta_2/\beta_1 \approx 2{\cdot}5$, $h \approx 8$ km, the first condition is satisfied if the convection velocity V is an order of magnitude less than 30 m sec^{-1} and the second if τ_0, the persistence time of eddy patterns in the boundary layer, is considerably larger than $\beta_1^{-\frac{1}{2}} = 70$ sec. Measurements in the laboratory indicate that the convection velocity of the large-scale pressure field at the surface relative to the free stream is roughly five times the friction velocity (e.g. Willmarth & Wooldridge 1962). In the atmosphere, the change from neutral stability at low heights to strong stability near the edge of the boundary layer may increase the ratio of convection velocity to friction velocity but the convection velocity is unlikely to exceed several metres per second. The life of a large eddy is expected to be of order the length-scale divided by the typical velocity fluctuation. In adiabatic conditions, we may estimate $L_0 = 300$ m and $w_0 = 0{\cdot}3$ m sec^{-1} leading to $\tau_0 \approx 1000$ sec. With strong convection, the time may be less. It appears that the analysis should give a reasonably accurate estimate of the rate of growth of internal waves in the absence of strong velocity gradients outside the boundary layer.

As a numerical example, suppose the root-mean-square vertical velocity to be $0{\cdot}3$ m sec^{-1}. Substituting in equation (4.6), we find

$$\zeta_m^2 = 0{\cdot}30t(1+z/h) \quad (t \text{ in seconds, } \zeta_m \text{ in metres}),$$

assuming $\beta_1^{\frac{1}{2}}\tau_1 \approx \beta_1^{\frac{1}{2}}\tau_0$ to be large. In the most favourable circumstances, for $\beta_1 L_0^2/V^2$ near $\frac{1}{8}\pi(1+2^{\frac{1}{2}}) = 0{\cdot}95$, wave displacements of order 100 m could appear after 3 h. With an integral scale of 300 m, the optimum convection velocity is 4 m sec^{-1}. Substituting in equation (4.13), the maximum value of

$$\left\langle\left(\frac{\partial\zeta}{\partial z}\right)^2\right\rangle \quad \text{is} \quad 9{\cdot}1\times 10^{-7}t \quad (t \text{ in seconds}),$$

and occasional values of $\partial\zeta/\partial z$ near -1 are likely to occur only after periods of excitation so long that the assumption of steady conditions in the atmosphere is highly implausible.

5. Discussion

The analysis of the second section applies to the generation of internal waves in a stably stratified fluid which is initially at rest by travelling disturbances of the kind found in a boundary layer. In general, most of the wave-energy appears in modes which satisfy a resonance condition that their phase velocity is nearly equal to the convection velocity of the disturbances in the boundary layer. If the fluid is in two layers, each of constant density gradient with the more stable on top, the generated waves take the form of groups with crests at right-angles to the vector difference between the convection velocity of the disturbances and the general velocity of the fluid. The wavelength in the group is proportional to the vector difference and the lateral extent and appearance of the groups depends on the ratio of the wavelength to the length-scale of the disturbances, or $V/(\beta_1^{\frac{1}{2}} L_0)$. If the ratio is large, consideration of the angle integral in equation (4·3) shows that wave-energy is concentrated in wave-numbers with components $\beta_1^{\frac{1}{2}}/V$ in the direction of $\mathbf{V}$ and roughly $\pm L_0^{-1}$ at right-angles. The effect is that the wave-groups have a spanwise modulation of amplitude, typically a reversal of sign over a distance of order L_0. If the ratio is small, the groups are simple. The vertical distribution of wave-amplitude is not known with any certainty. The reason is that the low-order modes which contribute most to the amplitude at considerable heights may cause resonance effects in the upper layer which depend critically on the assumed depth. Arguments given in § 3 suggest that the average distribution over the small range of depths sufficient to provide all degrees of resonance is one which increases moderately from the surface to the layer junction.

If numerical values appropriate to the atmosphere are substituted in the equation for the growth of waves, vertical displacements of order 100 m are predicted after several hours, and it seems possible that some occurrences of travelling wave-clouds may arise in this way. On the other hand, it is most unlikely that the wave-motion would ever become sufficiently intense to produce patches of clear-air turbulence by overturning. A serious restriction on the atmospheric application is that there should be no appreciable shear outside the boundary layer, but considerable difficulties arise in the presence of wind-shear.

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